THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Tutorial 6
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1. (Exercise 3.4.12 of [BS11]) Show that if $\{x_n\}$ is unbounded, then there exists a subsequence $\{x_{n_k}\}$ such that $\lim \left(\dfrac{1}{x_{n_k}}\right) = 0$.

2. (Exercise 3.4.14 of [BS11]) Suppose $\{x_n\}$ is a sequence which is bounded from above. Let $s = \sup\{x_n\}$. Show that either $s = x_N$ for some $N \in \mathbb{N}$ sufficiently large, or that there is a subsequence x_{n_k} so that $x_{n_k} \to s$ as $k \to +\infty$.

3. (Exercise 3.4.15 of [BS11]) Let $\{I_n := [a_n, b_n]\}$ be a nested sequence of closed bounded intervals. For each $n \in \mathbb{N}$, let $x_n \in I_n$. Use the Bolzano-Weierstrass Theorem to prove the Nested Intervals Theorem.

2. (Exercise 3.4.12 of [BS11]) Show that if $\{x_n\}$ is unbounded, then there exists a subsequence $\{x_{n_k}\}$ such that $\lim \left(\frac{1}{x_{n_k}}\right) = 0$.

Pf: Construct inductively. Since $\{x_n\}$ is unbounded, for all $M>0$, can find an $n \in \mathbb{N}$ s.t. $|x_n| > M$.

Set $M=1$. Then find $n_1 \in \mathbb{N}$ s.t. $|x_{n_1}| > 1$.

Set $M=2$: $\{x_n\}_{n \geq n_1}$ is still an unbounded sequence

Then take $n_2 \in \mathbb{N}$ s.t.

$$|x_{n_2}| > \max\{2, |x_1|, |x_2|, \dots, |x_{n_1}|\}.$$

Ensuring that $|x_{n_1}| \leq |x_{n_2}|$ and $|x_{n_2}| > 2 \Leftrightarrow \frac{1}{|x_{n_2}|} < \frac{1}{2}$.

Likewise for $M=k$, $\{x_n\}_{n \geq n_{k-1}}$ is still an unbounded sequence and can take $n_k \in \mathbb{N}$ s.t.

$$|x_{n_k}| > \max\{k, |x_1|, |x_2|, \dots, |x_{n_{k-1}}|\}$$

so that $|x_{n_{k-1}}| \leq |x_{n_k}|$, $n_{k-1} \leq n_k$, and $\frac{1}{|x_{n_k}|} < \frac{1}{k}$.

and clearly x_{n_k} s.t. $\lim\limits_{k \to \infty} \frac{1}{x_{n_k}} = 0$.

3. (Exercise 3.4.14 of [BS11]) Suppose $\{x_n\}$ is a sequence which is bounded from above. Let $s = \sup\{x_n\}$. Show that either $s = x_N$ for some $N \in \mathbb{N}$ sufficiently large, or that there is a subsequence x_{n_k} so that $x_{n_k} \to s$ as $k \to +\infty$.

Pf: Sps $x_n < s$ for any $n \in \mathbb{N}$. Then we construct the desired subsequence inductively with

$\{x_{n_k}\}$ s.t. $x_{n_k} > s - \frac{1}{k}$.

Take $\varepsilon = 1$. Then pick $n_1 \in \mathbb{N}$ s.t. $x_{n_1} > s - 1$. b/c $s = \sup\{x_n\}$

Sps $x_{n_1}, \dots, x_{n_k}$ exist s.t. $x_{n_l} > s - \frac{1}{l}$ for $l = 1, \dots, k$

n_{k+1} s.t. $x_{n_{k+1}} > s - \frac{1}{k+1}$

Need to guarantee that $n_{k+1} > n_k$

If we can show that $s = \sup\{x_n : n > n_k\}$, then we can freely take n_{k+1} s.t. $x_{n_{k+1}} > s - \frac{1}{k+1}$ and $n_{k+1} > n_k$.

$\sup\{x_n : n > n_k\} \le \{x_n\}$, so we have

$V := \sup\{x_n : n > n_k\} \le s$.

Sps $v < s$. Then since we have $x_n < s$ by assumption, restricting to $n = 1, \dots, n_k$, we have

$\max\{v, \max\{x_n : n = 1, \dots, n_k\}\} < s$

But $\{x_n : n > n_k\} \cup \{x_n : n = 1, \dots, n_k\} = \{x_n\}$

So we have a contradiction to the fact that $s = \sup\{x_n\}$

So we have that $v = s$. //

4. (Exercise 3.4.15 of [BS11]) Let $\{I_n := [a_n, b_n]\}$ be a nested sequence of closed bounded intervals. For each $n \in \mathbb{N}$, let $x_n \in I_n$. Use the Bolzano-Weierstrass Theorem to prove the Nested Intervals Theorem.

Pf: We have a sequence $\{x_n\}$ s.t. $x_n \in I_n$ for each n.
Then since the intervals are nested, our sequence is bounded, $a_1 \leq x_n \leq b_1$. So by BW theorem, $\{x_n\}$ admits a convergent subsequence, with limit, say L.
WTS $L \in \bigcap_{k=1}^{\infty} I_k$, i.e. WTS $L \in I_k$ for each $k \in \mathbb{N}$.
Let k be fixed. Then for $m > k$ there are corresponding $n_k \leq n_m$ s.t.

$$a_k \leq x_{n_m} \leq b_k$$

Then taking $m \to \infty$, we have $a_k \leq L \leq b_k$. //

The uniqueness part is the same as in original proof.